Arithmetical closures of a hierarchy of prewellorderings under AD, part II

[Convention: Use $\mathbb{R}$ to denote ω^ω]

Wadge hierarchy

0: $\{\emptyset\}$, $\{\mathbb{R}\}$

1: $\Delta^0_1 \setminus \{\emptyset, \mathbb{R}\}$

2: $\Sigma^0_1 \setminus \Pi^0_1$, $\Pi^0_1 \setminus \Sigma^0_1$

3: $\Delta(\mathrm{Diff}(2, \Sigma^0_1)) \setminus$ without what came before

4: $\mathrm{Diff}(2, \Sigma^0_1) \setminus \check{\mathrm{Diff}}(2, \Sigma^0_1)$, $\check{\mathrm{Diff}}(2, \Sigma^0_1) \setminus \mathrm{Diff}(2, \Sigma^0_1)$

$$\mathrm{Diff}(2, \Sigma^0_1) = \{A \cap B \mid A \in \Sigma^0_1, B \in \Sigma^0_1\}$$

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$\{\emptyset\}$ Σ^0_1 Σ^0_2 Σ^0_3

Δ^0_1 Δ^0_2 Δ^0_3

$\{\mathbb{R}\}$ Π^0_1 Π^0_2 Π^0_3

0 1 2 $\cdots\cdots$ ω_1 $\cdots$ $\omega_1^{\omega_1}$

Assume AD and DC from now on.

A norm is a surjective function $\varphi : \mathbb{R} \twoheadrightarrow \alpha$.

There is a bijective correspondence between norms and prewellorderings of $\mathbb{R}$.

prewellorderings: preorderings $\leq$ s.t. $(\mathbb{R}/\equiv, \leq)$ is a wellordering

$x \equiv y \iff x \leq y \wedge y \leq x$

How to get from norm to pwo:

$$x, y \in \mathbb{R}: \quad x <_\varphi y \iff \varphi(x) < \varphi(y).$$

! !

Def: φ, ψ are norms.

(1) $\varphi \leq_N \psi :\Longleftrightarrow \exists f: \mathbb{R} \longrightarrow \mathbb{R}$ continuous s.t. $\forall x \in \mathbb{R}: \varphi(x) \leq \psi(f(x))$

(2) $\varphi \leq_{NL} \psi :\Longleftrightarrow \exists g: \mathbb{R} \longrightarrow \mathbb{R}$ Lipschitz s.t. $\forall x \in \mathbb{R}: \varphi(x) \leq \psi(g(x))$.

For $\varphi, \psi: \mathbb{R} \longrightarrow X$ (any set) and $R \subseteq X \times X$ define games $G_W^R(\varphi, \psi)$ and $G_L^R(\varphi, \psi)$.

$G_L^R(\varphi, \psi)$:

$$\begin{array}{llllll} \mathrm{I} & x_0 & & x_1 & \dots & \rightsquigarrow x \in \mathbb{R} \\ \mathrm{II} & & y_0 & & y_1 & \rightsquigarrow y \in \mathbb{R} \end{array}$$

II wins $\Longleftrightarrow \varphi(x) \, R \, \psi(y)$.

G_W^R differs only by allowing II to pass. Player II loses there if she passes eventually.

<u>Prop:</u> $\varphi \leq_N \psi \iff$ Player II wins $G_W^{\leq}(\varphi, \psi)$.

$\varphi \leq_{NL} \psi \iff$ — " — $G_L^{\leq}(\varphi, \psi)$.

<u>Theorem:</u> Let N be the set of norms.

Then $(N/\equiv_N, \leq_N)$, $\underline{(N/\equiv_{NL}, \leq_{NL})}$ are <u>well-orders.</u>

<u>Proof:</u>

Linearity: $\varphi \not\leq_{NL} \psi \overset{AD}{\iff}$ I wins $G_L^{\leq}(\varphi, \psi)$

$\Longrightarrow$ II wins $G_L^{\not\geq}(\psi, \varphi)$

$\Longrightarrow$ II wins $G_L^{\leq}(\psi, \varphi)$

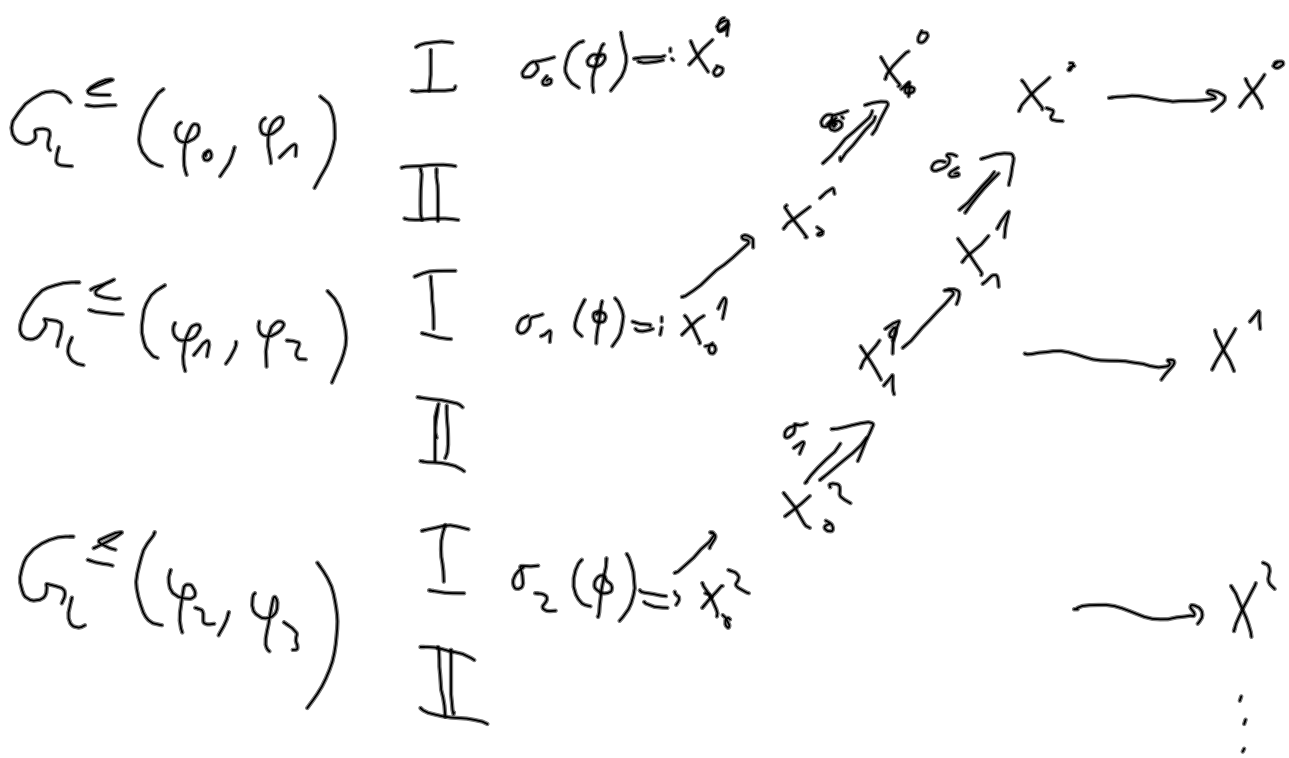
$\Longrightarrow$ $\psi \leq_{NL} \varphi$.

Well-foundedness

Assume $\langle \varphi_n \mid n \in \omega \rangle$ was $<_{NL}$-descending sequence.
Then I wins all games of the form

$$G_L^{\leq}(\varphi_n, \varphi_{n+1}).$$

Choose w.s. σ_n for I in $G_L^{\leq}(\varphi_n, \varphi_{n+1})$ and play all these games simultaneously as follows:



By σ_n winning for I we get

$$\forall n \quad \varphi_{n+1}(x^{n+1}) < \varphi_n(x^n). \quad ↯ \qquad \boxtimes$$

Q: What is $\Sigma := \mathrm{otyp}(N/{\equiv_N}, \leq_N)$

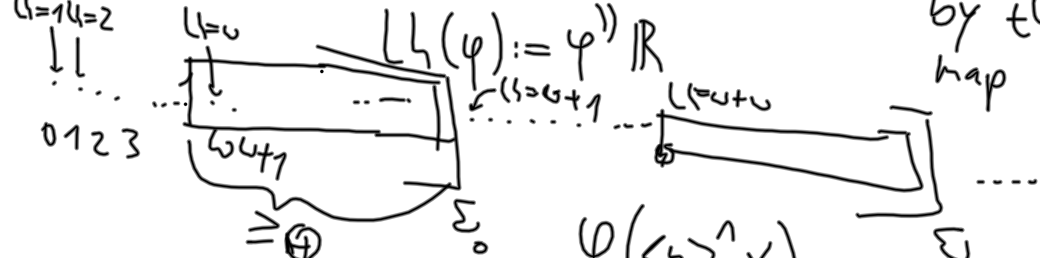
hierarchy of norms

Thm: (Löwe) $\Theta^2 \leq \Sigma < \Theta^+$

Thm: (B.) $\Theta^{(\Theta^\Theta)} \leq \Sigma$

$\varphi(x) = 1$, ψ s.t. $\psi''\mathbb{R} \subseteq 2$ then $\psi \leq_N \varphi$ is witnessed by the constant map $y \longmapsto x$.

$LL(\varphi) := \varphi''\mathbb{R}$



$\varphi(\langle n\rangle^\frown x) = n$

<u>Lemma:</u> $\varphi, \psi \in N$

(1) $\mathrm{lh}(\varphi) < \mathrm{lh}(\psi) \Rightarrow \varphi <_{NL} \psi$

(2) $\mathrm{lh}(\varphi) = \mathrm{lh}(\psi) = \alpha + 1 \Rightarrow \varphi \equiv_{NL} \psi$.

<u>Def:</u> Let $\langle \lambda_\alpha \mid \alpha < \Theta \rangle$ be the enumeration of all limit ordinals below Θ.

Then let for $\alpha < \Theta$:

$$\Sigma_\alpha := \sup \{ |\varphi|_N \mid \mathrm{lh}(\varphi) \leq \lambda_\alpha \}.$$

<u>Fact:</u> $\Sigma = \sup_{\alpha < \Theta} \Sigma_\alpha$.

Theorem (Löwe)

$\forall \alpha < \Theta \left(\Sigma_\alpha \geq \Theta \cdot \alpha \right)$ and $\Theta \leq \Sigma_0$.

Goal: Show that all Σ_α's are closed under ordinal multiplication, i.e.,

$$\forall \beta, \gamma < \Sigma_\alpha : \beta \cdot \gamma < \Sigma_\alpha .$$

For Wadge hierarchy we have an operation $\boxtimes : \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ s.t. for all self-dual $A \in \mathcal{P}(\mathbb{R})$ and any $B \in \mathcal{P}(\mathbb{R})$ s.t. $|B|_W$ is limit w. $\mathrm{cf}(|B|_W) > \omega$, we have:

$$|A \boxtimes B|_W = |A|_W \cdot \omega_1 \cdot |B|_W$$

<u>Problem 1:</u> Natural adaptation of $\otimes$ to norms may destroy surjectivity.

<u>Problem 2:</u> What is natural notion of s.d. for Ho N.

<u>ad 1:</u> Consider $\mathcal{N}' := \{\varphi : \mathbb{R} \longrightarrow \Theta \mid \varphi''\mathbb{R} \text{ bounded in } \Theta\}$.

Then extend $\leq_N$, $\leq_{NL}$ naturally.

<u>Prop:</u> $(\mathcal{N}'/\equiv_N, \leq_N) \cong (\mathcal{N}/\equiv_N, \leq_N)$.

We call elements of $\mathcal{N}'$ also <u>weak norms</u>.

ad 2) For $A \subseteq \mathbb{R}$ we had

$$A \text{ s.d.} \iff A \leq_w \mathbb{R} \setminus A$$
$$\iff \mathrm{II} \text{ wins } G_w^{=}(\chi_A, \chi_{\mathbb{R}\setminus A})$$
$$\iff \mathrm{II} \text{ wins } G_w^{\neq}(\chi_A, \chi_A)$$
$$\iff \mathrm{II} \text{ wins } G_w^{\not>}(\chi_A, \chi_A)$$

, considering $\{0,1\}$ as antichain of length 2.

For φ a norm:

$$\varphi \text{ s.d.} \iff \mathrm{II} \text{ wins } G_w^{<}(\varphi, \varphi).$$